MF4 : BILAN DE QUANTITÉ DE MOUVEMENT ET DE MOMENT CINÉTIQUE

I_ Bilans de quantité de mouvement

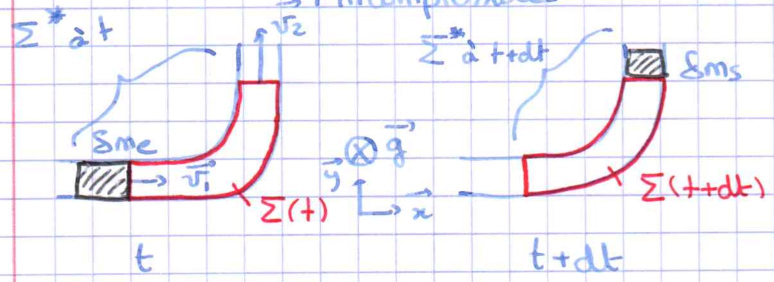
1_ Bilans de quantité de mouvement _ Démarche générale

- on doit se ramener à un sys. fermé → 2 dessins de Σ^* à t et t+dt
- Bilan sur Σ^* : $\vec{p}^*(t) = \dots$ et $\vec{p}^*(t+dt) = \dots$
- PFD à Σ^* :

$$\frac{d\vec{p}^*}{dt} = \sum \vec{F}^{ext}$$

2_ Ex. 1 : force subie par un tuyau coudé

Hypos :
- tuyau de section cste
- éc^t | stat
  | pft (pas de perte de charge)
  | incompressible



- Bilan de q^{té} de mvm^t sur Σ^*

$$\vec{p}^*(t) = \vec{p}(t) + \delta m_e \vec{v_1} = \vec{P}(t) + D_m dt\, \vec{v_1}$$
$$\vec{p}^*(dt+t) = \vec{p}(t+dt) + \delta m_s \vec{v_2} = \vec{P}(t) + D_m dt\, \vec{v_2}$$

$$\Rightarrow \frac{d\vec{P}^*}{dt} = D_m [\vec{v_2} - \vec{v_1}]$$

- éc^t incomp : $D_v = c^{te}$ donc $v_1 S = v_2 S \Rightarrow v_1 = v_2$

$$\Rightarrow \frac{d\vec{p}^*}{dt} = D_m v_1 (\vec{u_y} - \vec{u_x}) = \mu S v^2 (\vec{u_y} - \vec{u_x})$$

- PFD à Σ^* :

$$\frac{d\vec{p}^*}{dt} = \sum \vec{F}^{ext} = m\vec{g} + \underbrace{\vec{F}_{t\to eau}}_{= -\vec{F}_{eau\to t}} + P_1 S \vec{u_x} - P_2 S \vec{u_y}$$

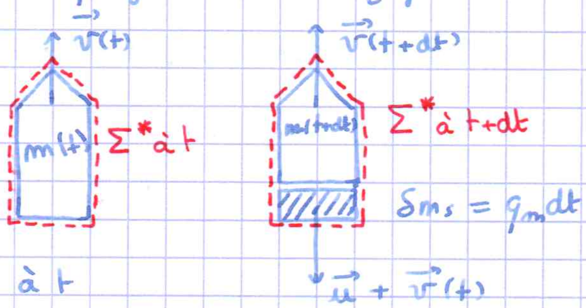
→ Bernoulli entre entrée et sortie :

$$P_1 + \mu \frac{v_1^2}{2} = P_2 + \mu \frac{v_2^2}{2} \Rightarrow P_2 = P_1 \text{ donc } \vec{F}_{eau\to t} = m\vec{g} + (P_1 + \mu v^2) S (\vec{u_x} - \vec{u_y})$$

3_ Tir vertical d'Ariane 5 _ Poussée

1/ $q_m = \dfrac{m_p}{\Delta t} = \dfrac{520}{130} = 4 \text{ t/s}$.

2/ • sys fermé $\Sigma^* = \{$ fusée + contenu à $t \}$ de masse $m(t)$



• Bilan de qd de mvt sur Σ^*

⚠ on n'est pas en R.P.

$$\vec{p}^{\,*}(t) = m(t)\,\vec{v}(t)$$

$$\vec{p}^{\,*}(t+dt) = m(t+dt)\,\vec{v}(t+dt) + q_m dt\,(\vec{u} + \vec{v}(t))$$

$$\Rightarrow \frac{d\vec{p}^{\,*}}{dt} = \frac{d}{dt}\left(m(t)\vec{v}(t)\right) + q_m(\vec{u} + \vec{v}(t))$$

$$= \frac{dm}{dt}\vec{v}(t) + m(t)\frac{d\vec{v}}{dt} + q_m(\vec{u} + \vec{v}(t))$$

or $m(t+dt) + q_m dt = m(t) \Rightarrow \dfrac{dm}{dt} = -q_m$

donc $\dfrac{d\vec{p}^{\,*}}{dt} = m(t)\dfrac{d\vec{v}}{dt} + q_m\vec{u}$

• PFD à Σ^* : $\dfrac{d\vec{p}^{\,*}}{dt} = \sum \vec{F}^{ext} = m(t)\vec{g}$

donc $m(t)\dfrac{d\vec{v}}{dt} + q_m\vec{u} = m(t)\vec{g}$

$$m(t)\frac{d\vec{v}}{dt} = m(t)\vec{g} + \vec{\Pi} \qquad \vec{\Pi} = -q_m\vec{u}$$

• Décollage ?

si $\underbrace{q_m u}_{12\text{ MN}} > \underbrace{m_0 g}_{7{,}8\text{ MN}} \Rightarrow$ oui

3/ • vitesse $v(t)$? $\left| \begin{array}{l} \vec{v}(t) = v(t).\vec{u_z} \\ \vec{u} = -u.\vec{u_z} \end{array} \right.$

$m(t) = m_0 - q_m t$

$$m(t)\frac{dv}{dt} = -m(t)g + q_m u$$

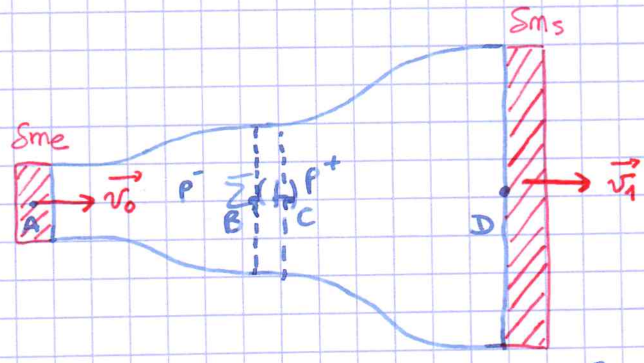
$$\Rightarrow dv = -g\,dt + \frac{q_m\,dt}{m_0 - q_m t}u$$

$$v(t) - v(0) = -gt - u \ln\left(\frac{m_0 - q_m t}{m_0}\right)$$

$$v(t) = -gt - u \ln\left(1 - \frac{q_m t}{m_0}\right)$$

$$z(t) = \frac{u m_0}{q_m}\left(1 - \frac{q_m t}{m_0}\right)\left[\ln\left(1 - \frac{q_m t}{m_0}\right) - 1\right] + \frac{u m_0}{q_m} - \frac{g t^2}{2}$$

4 _ Ex 3 : Rendement maximal d'une éolienne



① Bilan de $\vec{p}$:

$$\vec{P}^*(t) = \vec{p}(t) + \delta m_e \vec{v_0}$$

$$\vec{P}^*(t+dt) = \vec{p}(t+dt) + \delta m_s \vec{v_1}$$

$$\frac{d\vec{P}^*}{dt} = D_m(\vec{v_1} - \vec{v_0})$$

② PFD à Σ^* : $\displaystyle \frac{d\vec{P}^*}{dt} = \vec{F}_{h \to a} + \cancel{m\vec{g}} + \underbrace{\vec{R}^{ext}_{pres}}_{\vec{0} \text{ car } P_0 \text{ uniforme en tt point de la surface FERMÉE}}$

$$\Rightarrow \vec{F} = D_m(\vec{v_0} - \vec{v_1})$$

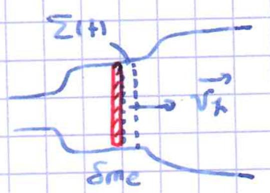
On ne peut appliquer Bernoulli entre "−" et "+" car hélice prélève de l'énergie.

→ entre A et B : $P_0 + \mu \frac{v_0^2}{2} = P^- + \mu \frac{v_h^2}{2}$

→ entre C et D : $P^+ + \mu \frac{v_h^2}{2} = P_0 + \mu \frac{v_1^2}{2}$

$$\Rightarrow P^- - P^+ = \mu\left(\frac{v_0^2}{2} - \frac{v_1^2}{2}\right)$$

③ $\vec{F}$ en fonction de $P^- - P^+$? → Bilan de $\vec{p}$ sur Σ^*



$$\frac{d\vec{p}^*}{dt} = D_m(\vec{v_h} - \vec{v_h}) = \vec{0}$$

$$\vec{0} = -\vec{F} + P^- S_h \vec{u_x} - P^+ S_h \vec{u_x}$$

$$\vec{F} = (P^- - P^+) S_h \vec{u_x} \quad \text{or} \quad \vec{F} = D_m(v_0 - v_1)\vec{u_x}$$

$$\Rightarrow \frac{\mu}{2}(v_0^2 - v_1^2) S_h = \mu v_h S_h (v_0 - v_1) \quad \Rightarrow \quad v_h = \frac{v_0 + v_1}{2}$$

donc $\vec{F} = 2\mu v_h S_h (v_0 - v_h)\,\vec{u_x}$

Bilan d'E_c sur le "grand" Σ^* :

$$E_c^*(t) = E_c(t) + D_m\,dt \times \frac{v_0^2}{2}$$

$$E_c^*(t+dt) = E_c(t+dt) + D_m\,dt \times \frac{v_1^2}{2}$$

$$\frac{dE_c^*}{dt} = D_m\left[\frac{v_1^2}{2} - \frac{v_0^2}{2}\right]$$

$$\frac{dE_c^*}{dt} = P^{ext} + \underbrace{P^{int}}_{P^{visc} + P^{int}_{press}}$$ or incomp + parfait donc $P^{int} = 0$

$$= \underbrace{P_{h\to a}}_{-P} + \underbrace{P^{ext}_{press}}_{\substack{\shortparallel \\ 0\ (incomp)}} \Rightarrow P = -\frac{dE_c^*}{dt} = D_m\left(\frac{v_0^2}{2} - \frac{v_1^2}{2}\right)$$

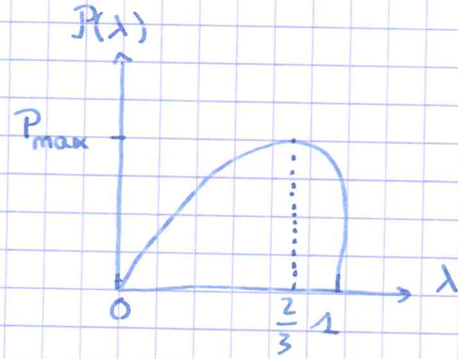
$$P = \frac{\mu v_h S_h}{2}(v_0+v_1)(v_0-v_1)$$

$$= 2\mu S_h v_h^2 (v_0 - v_h)$$

$$= \vec{F}\cdot\vec{v_h}$$

$$\lambda = \frac{v_h}{v_0} = \frac{v_0+v_1}{2v_0} < 1 \qquad \lambda < 1$$

$$P(\lambda) = 2\mu S_h v_0^3 \lambda^2 (1-\lambda)$$



$$P_{max} = 2\mu S_h v_0^3 \frac{4}{9} \times \frac{1}{3} = \frac{8}{27}\mu S_h v_0^3$$

$$r = \frac{P}{D_m \frac{v_0^2}{2}} = \frac{2\mu S_h v_0^3 \lambda^2(1-\lambda)}{\mu S_h \frac{v_0^3}{2}}$$

$$= 4\lambda^2(1-\lambda) \Longrightarrow r_{max} = \frac{16}{27} = 59\ \%$$ limite de Betz

Eolienne "réseau"

$$P_{max} = \frac{8}{27}\mu \underbrace{S_h}_{\frac{\pi D^2}{4}} v_0^3 \quad \# \ 2MW\ !$$

II _ Bilan de moment cinétique

1_ Bilan de moment cinétique pour un sys. fermé en rotation autour d'un axe fixe Δ

(i) déf. le bon sys FERMÉ Σ^*
(ii) Bilan de moment cinétique

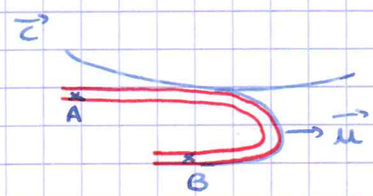
$$\left.\begin{array}{l} L^*_\Delta(t) = \dots \\ L^*_\Delta(t+dt) = \dots \end{array}\right| \Rightarrow \frac{dL^*_\Delta}{dt} = \dots$$

(iii) TMC à Σ^* : $\dfrac{dL^*_\Delta}{dt} = \mathcal{M}^{ext}_\Delta$

2_ Ex : fonctionnement d'une turbine Pelton

→ Centrales hydroélectriques

$C \# \sqrt{2gH}$ $\qquad c = 74\ m/s \Rightarrow H = 280\ m$



1/ R_{auget} : stat

Hypos dans R_{auget} : stat + incomp + pft

2/ cf. cours : $P = P°$ en # pts.

3/ ① Ds R_{auget} : $\vec{v}_A' = (c-u)\vec{u_x}$

$$P_A + \mu \frac{v_A'^2}{2} + \mu g z_A = P_B + \mu \frac{v_B'^2}{2} + \mu g z_B$$

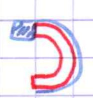
$$v_B' = v_A'$$

$\Rightarrow \vec{v}_{B'} = -(c-u)\vec{u_x}$

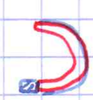
② Ds R_{terr} : $\vec{v_A} = c\vec{u_x}$

$\vec{v_B} = \vec{v}_B' + u.\vec{u_x} = (2u-c)\vec{u_x}$

4/



Σ^* à t



Σ^* à $t+dt$

$\vec{P}^*(t) = \vec{P}(t) + D_m dt\, c\, \vec{u_x}$

$\vec{P}^*(t+dt) = \vec{P}(t+dt) + D_m dt(2u-c)\vec{u_x}$

$\Rightarrow \dfrac{d\vec{p}^*}{dt} = D_m(2u-2c)\vec{u_x} = -2\rho Sc(c-u)\vec{u_x}$

② PFD à Σ^*

$$\frac{d\vec{p}^*}{dt} = \vec{F}_{aug \to eau} + \vec{R}_{P_0}$$

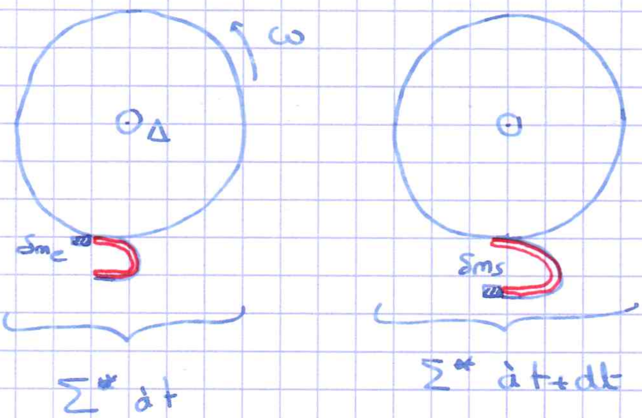
→ autre sys : {auget seul}

$$\vec{F}_{tot} = \vec{F}_{jet \to aug} + \vec{F}_{air} = -\vec{F}_{aug \to jet} + \vec{F}_{air}$$

$$= 2\rho Sc\,(c-u)\,\vec{u_x} + \underbrace{\vec{R_p} + \vec{F_{air}}}_{\text{surface fermée} \Downarrow = 0}$$

$$\vec{F}_{tot} = 2\rho Sc(c-u)\,\vec{u_x}$$

5/ Bilan de moment cinétique



Bilan de mom^t cinétiq $\quad L_\Delta^*(t) = L_\Delta(t) + R\,\delta m_e c$

$$L_\Delta^*(t+dt) = L_\Delta(t+dt) - R\,\delta m_s\,(c-2u)$$

or $L_\Delta(t) = J\omega = c^{ste}$

$$\frac{dL_\Delta^*}{dt} = D_m R\,(2u-c-c) = -2\rho Sc R(c-u)$$

TMC : $\dfrac{dL_\Delta^*}{dt} = \underbrace{-\Gamma_c}_{\text{couple résistant exercé par la charge}} + \underset{\uparrow\ p^o\ \text{unif}}{0} + \underset{\uparrow\ \text{pivot idéal}}{0} + 0_{poids}$

$$\Gamma_c = -\frac{dL_\Delta^*}{dt} = 2\rho SRc(c-u) > 0$$

Sys $\begin{Bmatrix}\text{rotor}\\\text{avec}\\\text{auget}\end{Bmatrix}$ $\quad \dfrac{d}{dt}(J\omega) = 0 = -\Gamma_c + \Gamma_{jet \to rotor} \Rightarrow \Gamma = \Gamma_c$

6/ Bilan d'énergie cinétique $\quad$ m̂ Σ^* qu'en 5

$$E_c^*(t) = E_c(t) + D_m\,dt \times \frac{c^2}{2}$$

$$E_c^*(t+dt) = E_c(t+dt) + D_m\,dt \times \frac{(2u-c)^2}{2}$$

$E_c(t) \simeq \frac{1}{2} J\omega^2 = cste$

$$\frac{dE_c^*}{dt} = \frac{\rho S c}{2}\left((2u-c)^2 - c^2\right) = 2\rho S c u (u-c) < 0$$

TEC à Σ^* :

$$\frac{dE_c^*}{dt} = \mathcal{P}^{ext} + \underbrace{\mathcal{P}^{int}}_{=0} = \underbrace{\mathcal{P}^{c}} + \underbrace{\mathcal{P}^{ext}_{pres}}_{=0}$$

puissance fournie à l'alternateur

$\Rightarrow \mathcal{P}_c = \frac{dE_c^*}{dt}$

$\mathcal{P}_c = \rho S c u (u-c) < 0$

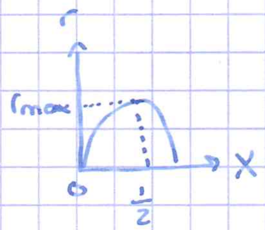
$\rightarrow$ {rotor seul} $\quad \frac{d}{dt}\left(\frac{1}{2} J\omega^2\right) = 0 = \mathcal{P}_c + \underbrace{\mathcal{P}_{j\to r}}_{\mathcal{P}}$

$\Rightarrow \mathcal{P} = -\mathcal{P}_c = 2\rho S c u (u-c) > 0$

NB : $\left\{ \begin{array}{l} \mathcal{P} = \vec{F} . u \vec{u_x} \\ \text{et} = \Gamma_x \omega \end{array} \right.$

7/ $r = \frac{\mathcal{P}}{\mathcal{P}_{inc}} = \frac{2\rho S c u (c-u)}{\underbrace{\rho S c}_{D_m} \times \frac{c^2}{2}} = \frac{4u(c-u)}{c^2}$

$X = \frac{u}{c}$: $\quad r(X) = 4X(1-X)$



$r_{max} = r\left(\frac{1}{2}\right) = 1$ obtenu pour $u = \frac{c}{2}$

$\frac{c}{2} = R\omega \Rightarrow R = \frac{c}{2\omega} = \frac{74}{2 \times \frac{7500\pi}{60}} = 47\,cm$

$\left\{ \begin{array}{l} \mathcal{P}_{max} = P_{inc} = \frac{\rho S c^3}{2} = 4,1\,MW. \\ \mathcal{P}_{réel} = r\mathcal{P}_{max} = 3,6\,MW. \end{array} \right.$